

# Fixed Points of Augmented Generalized Happy Functions

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## Abstract

An augmented generalized happy function  $S_{[c,b]}$  maps a positive integer to the sum of the squares of its base  $b$  digits plus  $c$ . In this paper, we study various properties of the fixed points of  $S_{[c,b]}$ ; count the number of fixed points of  $S_{[c,b]}$ , for  $b \geq 2$  and  $0 < c < 3b - 3$ ; and prove that, for each  $b \geq 2$ , there exist arbitrarily many consecutive values of  $c$  for which  $S_{[c,b]}$  has no fixed point.

## 1 Introduction

The concept of a happy number, defined in [5] and popularized by [3], was generalized in [2] by allowing for varying bases and exponents in the defining function. In [1], this was generalized further, altering the defining function with the addition of a constant. Specifically, for integers  $c \geq 0$  and  $b \geq 2$ , the augmented generalized happy function,  $S_{[c,b]} : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ , is defined by

$$S_{[c,b]} \left( \sum_{i=0}^n a_i b^i \right) = c + \sum_{i=0}^n a_i^2, \quad (1)$$

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where  $0 \leq a_i \leq b - 1$  and  $a_n \neq 0$ . Thus, for a positive integer  $a$  denoted  $a_n \cdots a_1 a_0$  in base  $b$ ,

$$S_{[c,b]}(a_n \dots a_1 a_0) = c + a_n^2 + \dots + a_1^2 + a_0^2.$$

A positive integer  $a$  is a happy number if for some  $k \in \mathbb{Z}^+$ ,  $S_{[0,10]}^k(a) = 1$ . Although 1 is the sole fixed point of  $S_{[0,10]}$ , as shown in [2], for  $b \neq 10$ ,  $S_{[0,b]}$  may have additional fixed points. Similarly, as shown in [1], when  $c > 0$  (and  $b \geq 2$ ),  $S_{[c,b]}$  may have multiple fixed points.

In this work, we study the fixed points of the functions  $S_{[c,b]}$ . First, in Section 2, we prove some preliminary results providing properties of the fixed points and consecutive fixed points of an arbitrary, but fixed,  $S_{[c,b]}$ . Then, in Section 3, we discuss the exact number of fixed points of  $S_{[c,b]}$ , in terms of  $c$  and  $b$ . Finally, in Section 4, we let  $c \geq 0$  vary and prove that for each  $b \geq 2$ , there are arbitrarily long sequences of consecutive values of  $c$  for which  $S_{[c,b]}$  has no fixed point.

## 2 Fixed Point Characteristics

Here we discuss a variety of results concerning relationships between the fixed points of a single  $S_{[c,b]}$ , where  $c \geq 0$  and  $b \geq 2$  are arbitrary integers. The first theorem concerns consecutive fixed points of  $S_{[c,b]}$ .

**Theorem 2.1.** *Fix  $c \geq 0$  and  $b \geq 2$ .*

1. *If  $a \in \mathbb{Z}^+$  is a multiple of  $b$ , then  $a$  is a fixed point of  $S_{[c,b]}$  if and only if  $a + 1$  is a fixed point of  $S_{[c,b]}$ .*
2. *Every consecutive pair of fixed points of  $S_{[c,b]}$  has a multiple of  $b$  as its first member.*
3. *There is no consecutive triplet of fixed points of  $S_{[c,b]}$ .*

*Proof.* We begin with the proof of Part 1. Since  $a$  is a multiple of  $b$ , we have  $S_{[c,b]}(a + 1) = S_{[c,b]}(a) + 1^2 = S_{[c,b]}(a) + 1$ . Thus  $S_{[c,b]}(a) = a$  if and only if  $S_{[c,b]}(a + 1) = a + 1$ .

For Part 2, assume that  $a$  and  $a + 1$  are both fixed points of  $S_{[c,b]}$  and using standard notation for base  $b$ , let

$$a = \sum_{i=0}^n a_i b^i.$$

First, assume that  $a_0 \neq b - 1$ . Then

$$\begin{aligned} a &= S_{[c,b]}(a) = c + \sum_{i=1}^n a_i^2 + a_0^2, \quad \text{and so} \\ a + 1 &= S_{[c,b]}(a + 1) = c + \sum_{i=1}^n a_i^2 + (a_0 + 1)^2 \\ &= a + 2a_0 + 1. \end{aligned}$$

Thus,  $2a_0 = 0$ , implying that  $a_0 = 0$ . Therefore  $a$  is a multiple of  $b$ , as desired.

Next assume, for a contradiction, that  $a_0 = b - 1$ . Let  $j \in \mathbb{Z}^+$  be minimal such that  $a_j \neq b - 1$ . (If every digit of  $a$  is equal to  $b - 1$ , then  $a + 1 = b^k$  for some natural number  $k$  and  $S_{[c,b]}(a + 1) = 1$ , so  $a + 1$  is not a fixed point.) Then

$$a = S_{[c,b]}(a) = c + \sum_{i=j}^n a_i^2 + j(b - 1)^2, \quad (2)$$

and since

$$\begin{aligned} a + 1 &= \sum_{i=j+1}^n b^i a_i + (a_j + 1)b^j, \quad \text{we have} \\ a + 1 &= S_{[c,b]}(a + 1) = c + \sum_{i=j+1}^n a_i^2 + (a_j + 1)^2. \end{aligned} \quad (3)$$

Combining equations (2) and (3) yields

$$a_j^2 + j(b - 1)^2 + 1 = (a_j + 1)^2.$$

Thus,  $j(b - 1)^2 = 2a_j$ . Since  $a_j < b - 1$ ,  $j(b - 1) < 2$  and so  $j = b - 1 = 1$ . But then  $2a_j = 1$ , which is a contradiction.

Finally, Part 3 is immediate from Part 2.  $\square$

Lemma 2.2 provides another pairing of fixed points of  $S_{[c,b]}$ .

**Lemma 2.2.** *Fix  $c \geq 0$ ,  $b \geq 2$ , and  $a \in \mathbb{Z}^+$  where*

$$a = \sum_{i=0}^n a_i b^i,$$

in standard base  $b$  notation with  $a_1 \neq 0$ . Let

$$\tilde{a} = \sum_{i=2}^n a_i b^i + (b - a_1)b + a_0.$$

Then  $a$  is a fixed point of  $S_{[c,b]}$  if and only if  $\tilde{a}$  is a fixed point of  $S_{[c,b]}$ .

*Proof.* Assume that  $a$  and  $\tilde{a}$  are as above, and that  $a$  is a fixed point of  $S_{[c,b]}$ . Then

$$\begin{aligned} S_{[c,b]}(\tilde{a}) &= S_{[c,b]} \left( \sum_{i=2}^n a_i b^i + (b - a_1)b + a_0 \right) = c + \sum_{i=2}^n a_i^2 + (b - a_1)^2 + a_0^2 \\ &= c + \sum_{i=0}^n a_i^2 + b^2 - 2a_1b = S_{[c,b]}(a) + b^2 - 2a_1b = a + b^2 - 2a_1b \\ &= \sum_{i=0}^n a_i b^i + (b - 2a_1)b = \sum_{i=2}^n a_i b^i + (b - a_1)b + a_0 = \tilde{a}. \end{aligned}$$

Therefore  $\tilde{a}$  is also a fixed point of  $S_{[c,b]}$ . The converse is immediate by symmetry.  $\square$

Finally, we consider the parity of  $c$  that is required for  $S_{[c,b]}$  to have a fixed point.

**Lemma 2.3.** Fix  $c \geq 0$  and  $b \geq 2$ , and let  $a = \sum_{i=0}^n a_i b^i$  be a fixed point of  $S_{[c,b]}$ , in the usual base  $b$  notation.

1. If  $b$  is odd, then  $c$  is even.

2. If  $b$  is even, then  $c \equiv \sum_{i=1}^n a_i \pmod{2}$ .

*Proof.* For  $b$  odd, by [1, Lemma 2.3],  $S_{[c,b]}(a) \equiv c + a \pmod{2}$ , which implies that  $c$  is even. For  $b$  even, since  $a$  is a fixed point of  $S_{[c,b]}$ ,

$$a_0 \equiv a = S_{[c,b]}(a) = c + \sum_{i=0}^n a_i^2 \equiv c + \sum_{i=0}^n a_i \pmod{2}.$$

Subtracting  $a_0$  from both sides of the congruence yields the result.  $\square$

### 3 Counting the Number of Fixed Points

In this section, we consider the *number* of fixed points of the function  $S_{[c,b]}$  for fixed  $c \geq 0$  and  $b \geq 2$ . In Corollary 3.5, we provide a formula for the number of fixed points of  $S_{[c,b]}$  for all values of  $b$  and a range of values of  $c$ , depending on  $b$ .

We begin by determining the number of fixed points of  $S_{[c,b]}$  of the form  $ub^n$ , where  $0 < u < b$  and  $n \geq 0$ . To fix notation, for  $c \geq 0$ ,  $b \geq 2$ , and  $n \geq 0$ , let

$$\mathcal{F}_{[c,b]}^{(n)} = \{a = ub^n \mid 0 < u < b \text{ and } S_{[c,b]}(a) = a\}.$$

In the following three lemmas, we provide conditions under which  $\mathcal{F}_{[c,b]}^{(n)}$  assumes specified values.

**Lemma 3.1.** *Fix  $b \geq 2$ . For  $c > 0$ ,  $\mathcal{F}_{[c,b]}^{(0)}$  is empty, while  $\mathcal{F}_{[0,b]}^{(0)} = \{1\}$ .*

*Proof.* Let  $0 < a < b$ . Then  $a = S_{[c,b]}(a)$  implies that  $c = a - a^2 \leq 0$ . Hence, if  $c = 0$ , we have  $a = 1$ , and if  $c > 0$ , we have a contradiction.  $\square$

**Lemma 3.2.** *Fix  $c \geq 0$  and  $b \geq 2$ . The cardinality of  $\mathcal{F}_{[c,b]}^{(1)}$  is*

$$\left| \mathcal{F}_{[c,b]}^{(1)} \right| = \begin{cases} 2 & \text{if } \alpha^2 - \alpha b + c = 0 \text{ for some integer } 1 \leq \alpha < \frac{1}{2}b, \\ 1 & \text{if } b^2 = 4c, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* From the definition of  $\mathcal{F}_{[c,b]}^{(1)}$ , given  $a = ub$  with  $0 < u < b$ , we have  $a \in \mathcal{F}_{[c,b]}^{(1)}$  if and only if  $S_{[c,b]}(a) = a$  or, equivalently,  $c + u^2 = ub$ . Thus,  $\left| \mathcal{F}_{[c,b]}^{(1)} \right| \neq 0$  if and only if there exists some integer  $u$ ,  $0 < u < b$ , such that

$$u^2 - ub + c = 0. \tag{4}$$

Since this is a quadratic equation, there are at most two such values of  $u$ , and at most one if  $b^2 = 4c$ .

By Lemma 2.2,  $a = ub$  is a fixed point of  $S_{[c,b]}$  if and only if  $\tilde{a} = (b - u)b$  is a fixed point of  $S_{[c,b]}$ . Hence  $a = ub \in \mathcal{F}_{[c,b]}^{(1)}$  if and only if  $(b - u)b \in \mathcal{F}_{[c,b]}^{(1)}$ . Thus,  $\left| \mathcal{F}_{[c,b]}^{(1)} \right| = 2$  if and only if there are two integer solutions to equation (4) with  $0 < u < b$ , in which case one of the solutions will satisfy  $1 \leq u < \frac{1}{2}b$ . The lemma follows.  $\square$

**Lemma 3.3.** For  $c \geq 0$ ,  $b \geq 2$ , and  $n \geq 2$ , the cardinality of  $\mathcal{F}_{[c,b]}^{(n)}$  is

$$|\mathcal{F}_{[c,b]}^{(n)}| = \begin{cases} 1 & \text{if } b^{2n} - 4c \text{ is a nonzero perfect square, and} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Fix  $n \geq 2$ , and suppose that  $a = ub^n$  is a fixed point of  $S_{[c,b]}$  for some  $0 < u < b$ . Then  $ub^n = a = S_{[c,b]}(a) = c + u^2$ , which implies that

$$u = \frac{b^n \pm \sqrt{b^{2n} - 4c}}{2}. \quad (5)$$

Since  $u \in \mathbb{Z}^+$ ,  $b^{2n} - 4c$  is a perfect square, and since  $u < b$  and  $n \geq 2$ ,  $b^{2n} - 4c$  is nonzero. Conversely, if  $b^{2n} - 4c$  is a nonzero perfect square, then  $\frac{b^n + \sqrt{b^{2n} - 4c}}{2} > b$  and so is not a candidate for  $u$ , while, letting  $u = \frac{b^n - \sqrt{b^{2n} - 4c}}{2}$ , it is easily verified that  $a = ub^n \in \mathcal{F}_{[c,b]}^{(n)}$ .  $\square$

The following theorem and its proof were inspired by the work of Hargreaves and Siksek [4] on the number of fixed points of (unaugmented) generalized happy functions. As is standard, we let  $r_2(n)$  denote the number of representations of  $n \in \mathbb{Z}^+$  as the sum of two squares; that is,

$$r_2(n) = |\{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = n\}|. \quad (6)$$

**Theorem 3.4.** For  $c > 0$  and  $b \geq 2$ , the number of two-digit fixed points of  $S_{[c,b]}$  is given by

$$\begin{cases} \frac{1}{2}r_2(b^2 - 4c + 1) + |\mathcal{F}_{[c,b]}^{(1)}| & \text{if } b \text{ is odd, and} \\ \frac{1}{4}r_2(b^2 - 4c + 1) + |\mathcal{F}_{[c,b]}^{(1)}| & \text{if } b \text{ is even.} \end{cases}$$

*Proof.* Note that  $a = ub + v$  is a fixed point of  $S_{[c,b]}$ , with  $0 < u < b$  and  $0 \leq v < b$  if and only if  $ub + v = S_{[c,b]}(ub + v) = c + u^2 + v^2$ . Define

$$U = \{(u, v) \in \mathbb{Z}^2 \mid 0 < u, v < b \text{ and } ub + v = c + u^2 + v^2\}.$$

By the correspondence  $(u, v) \leftrightarrow ub + v$ ,  $|U|$  is equal to the number of two-digit fixed points of  $S_{[c,b]}$  that are not multiples of  $b$ . Hence, the number of two-digit fixed points of  $S_{[c,b]}$  is equal to  $|U| + |\mathcal{F}_{[c,b]}^{(1)}|$ .

Set

$$X = \{(x, y) \in \mathbb{Z}^2 \mid y \geq 1 \text{ odd, and } x^2 + y^2 = b^2 - 4c + 1\}.$$

To see that  $|U| = |X|$ , consider the functions  $\phi : U \rightarrow X$  and  $\psi : X \rightarrow U$  defined by

$$\phi(u, v) = (2u - b, 2v - 1) \quad \text{and} \quad \psi(x, y) = \left( \frac{x + b}{2}, \frac{y + 1}{2} \right).$$

A straightforward calculation, and noting that  $2v - 1 > 0$  and odd, shows that the image of  $\phi$  is contained in  $X$ . Let  $(x, y) \in X$ , to see that  $\psi(x, y) \in U$ , first note that  $y$  is odd and  $x \equiv b \pmod{2}$ , and so  $\psi(x, y) \in \mathbb{Z}^2$ . Next, since  $x^2 < x^2 + y^2 = b^2 - 4c + 1 < b^2$ , we have  $-b < x < b$ , implying that  $0 < x + b < 2b$ , and thus  $0 < (x + b)/2 < b$ , as desired. Similarly,  $1 \leq y < b$ , so  $1 \leq (y + 1)/2 < b$ . Finally, a direct calculation verifies that the needed equation is satisfied.

Since, as is easily checked,  $\phi$  and  $\psi$  are inverses, it follows that  $|U| = |X|$ .

Now, note that  $X$  is a subset of

$$Z = \{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = b^2 - 4c + 1\},$$

and recall that, by equation (6),  $|Z| = r_2(b^2 - 4c + 1)$ .

If  $b$  is odd and  $(x, y) \in Z$ , then  $b^2 - 4c + 1 \equiv 2 \pmod{4}$ , and so  $y$  must be odd. Thus  $\varphi_{\text{odd}} : Z \rightarrow X$  defined by  $(x, y) \mapsto (x, |y|)$  is a 2-to-1 surjective function. Hence  $|X| = \frac{1}{2}|Z| = \frac{1}{2}r_2(b^2 - 4c + 1)$ .

If  $b$  is even and  $(x, y) \in Z$ , then  $b^2 - 4c + 1$  is odd, and so exactly one of  $x$  and  $y$  is odd. Thus  $\varphi_{\text{even}} : Z \rightarrow X$  defined by

$$(x, y) \mapsto \begin{cases} (x, |y|) & \text{if } y \text{ is odd,} \\ (y, |x|) & \text{if } x \text{ is odd,} \end{cases}$$

is a 4-to-1 surjective function. Hence  $|X| = \frac{1}{4}|Z| = \frac{1}{4}r_2(b^2 - 4c + 1)$ .

Recalling that the number of two-digit fixed points of  $S_{[c,b]}$  is  $|U| + |\mathcal{F}_{[c,b]}^{(1)}| = |X| + |\mathcal{F}_{[c,b]}^{(1)}|$ , the result follows.  $\square$

**Corollary 3.5.** *For  $b \geq 2$  and  $0 < c < 3b - 3$ , the number of fixed points of  $S_{[c,b]}$  is exactly*

$$\begin{cases} \frac{1}{2}r_2(b^2 - 4c + 1) + |\mathcal{F}_{[c,b]}^{(1)}| & \text{if } b \text{ is odd, and} \\ \frac{1}{4}r_2(b^2 - 4c + 1) + |\mathcal{F}_{[c,b]}^{(1)}| & \text{if } b \text{ is even.} \end{cases}$$

*Proof.* By [1, Lemma 2.2], since  $c < 3b - 3$ , for each  $a > b^2$ ,  $S(a) < a$  and, therefore,  $a$  is not a fixed point. Hence each fixed point of  $S_{[c,b]}$  has at most two digits. The corollary now follows directly from Lemma 3.1 and Theorem 3.4.  $\square$

## 4 Fixed Point Deserts

In this section, we fix the base  $b \geq 2$  and consider consecutive values of  $c$  for which  $S_{[c,b]}$  has no fixed points. Note that for a fixed  $b$ , if  $a = \sum_{i=0}^n a_i b^i$  is a fixed point of  $S_{[c,b]}$ , with  $0 \leq a_i < b$ , for each  $i$ , then, solving for  $c$ , we have

$$c = \sum_{i=0}^n a_i (b^i - a_i). \quad (7)$$

*Definition 1.* For  $b \geq 2$  and  $k \in \mathbb{Z}^+$ , an  $k$ -desert base  $b$  is a set of  $k$  consecutive non-negative integers  $c$  for each of which  $S_{[c,b]}$  has no fixed points. A desert base  $b$  is an  $k$ -desert base  $b$  for some  $k \geq 1$ .

For example, for  $28 \leq c \leq 35$ ,  $S_{[c,10]}$  has no fixed points and, therefore, there is an 8-desert base 10 starting at  $c = 28$ .

We begin by determining bounds on the values of  $c$  such that  $S_{[c,b]}$  has a fixed point of a given number of digits. For  $b \geq 2$  and  $n \geq 2$  define

$$\begin{aligned} m_{b,n} &= b^n - b^2 + 3b - 3, \text{ and} \\ M_{b,n} &= b^{n+1} - b^2 - (n-1)(b-1)^2 + (b - \lfloor b/2 \rfloor) \lfloor b/2 \rfloor. \end{aligned}$$

**Theorem 4.1.** *Let  $b \geq 2$  and  $n \geq 2$ . If  $S_{[c,b]}$  has a  $n+1$ -digit fixed point, then  $m_{b,n} \leq c \leq M_{b,n}$ . Further, these bounds are sharp.*

*Proof.* Let  $b \geq 2$  and  $n \geq 2$  be fixed. By equation (7), each fixed point  $a$  of  $S_{[c,b]}$  determines the value of  $c$ . Treating the  $a_i$  in equation (7) as independent variables taking on integer values between 0 and  $b-1$ , inclusive, we find the minimal possible value of  $c$  by minimizing each term. Observe that  $a_0(b^0 - a_0)$  is minimal when  $a_0 = b-1$ ; for  $0 < i < n$ ,  $a_i(b^i - a_i)$  is minimal when  $a_i = 0$ ; and, since  $a_n \neq 0$ ,  $a_n(b^n - a_n)$  is minimal when  $a_n = 1$ . Hence the minimal value of  $c$  is determined by

$$a = \sum_{i=0}^n a'_i b^i, \text{ where } a'_i = \begin{cases} 1, & \text{for } i = n; \\ 0, & \text{for } 1 \leq i \leq n-1; \\ b-1, & \text{for } i = 0, \end{cases}$$



and so, the minimal value of  $c$  is

$$c = (b-1)(b^0 - (b-1)) + 1 \cdot (b^n - 1) = b^n - b^2 + 3b - 3 = m_{b,n}.$$

Similarly, maximizing the terms of equation (7), we find that  $a_0(b^0 - a_0)$  is maximal when  $a_0 = 0$ ;  $a_1(b^1 - a_1)$  is maximal when  $a_1 = \lfloor b/2 \rfloor$ ; and for  $1 < i \leq n$ ,  $a_i(b^i - a_i)$  is maximal when  $a_i = b-1$ . Hence, the maximal value of  $c$  is determined by

$$a = \sum_{i=0}^n a_i'' b^i, \text{ where } a_i'' = \begin{cases} b-1, & \text{for } 2 \leq i \leq n; \\ \lfloor \frac{b}{2} \rfloor, & \text{for } i = 1; \\ 0, & \text{for } i = 0, \end{cases}$$

and, therefore, the maximal value of  $c$  is

$$\begin{aligned} c &= \lfloor b/2 \rfloor (b^1 - \lfloor b/2 \rfloor) + \sum_{i=2}^n (b^i - (b-1))(b-1) \\ &= b^{n+1} - b^2 - (n-1)(b-1)^2 + (b - \lfloor b/2 \rfloor) \lfloor b/2 \rfloor = M_{b,n}. \end{aligned} \quad \square$$

The following lemma is used to prove Theorem 4.3, which states that for each  $b \geq 2$  there exist arbitrarily long deserts base  $b$ .

**Lemma 4.2.** *Let  $b \geq 2$  and  $n \geq 2$ . Then, between the numbers  $M_{b,n}$  and  $m_{b,n+1}$ , there exists an  $k$ -desert base  $b$ , where*

$$k = m_{b,n+1} - M_{b,n} - 1 > (n - 5/4)(b-1)^2.$$

*Proof.* Let  $b \geq 2$  and  $n \geq 2$  be fixed. Note that

$$\begin{aligned} m_{b,n+1} - M_{b,n} - 1 &= (b^{n+1} - b^2 + 3b - 3) - \\ &\quad (b^{n+1} - b^2 - (n-1)(b-1)^2 + (b - \lfloor b/2 \rfloor) \lfloor b/2 \rfloor) - 1 \\ &\geq 3b - 3 + (n-1)(b-1)^2 - b^2/4 - 1 \\ &= (n-1)(b-1)^2 - b^2/4 + b/2 - 1/4 + 5b/2 - 15/4 \\ &> (n-1)(b-1)^2 - (b-1)^2/4 \\ &= (n - 5/4)(b-1)^2, \end{aligned}$$

since  $b \geq 2$ . Thus,

$$m_{b,n+1} > M_{b,n} + 1. \quad (8)$$

Recall that  $M_{b,n}$  is an upper bound on values of  $c$  such that  $S_{[c,b]}$  has a  $(n+1)$ -digit fixed point. Since  $M_{b,x}$  increases as  $x$  increases,  $M_{b,n}$  is an upper bound on values of  $c$  such that  $S_{[c,b]}$  has a fixed point with less than or equal to  $(n+1)$ -digits. Similarly,  $m_{b,n+1}$  is a lower bound on values of  $c$  such that  $S_{[c,b]}$  has a fixed point with greater than or equal to  $(n+2)$ -digits.

Thus, by equation (8), there is no value of  $c$  between  $M_{b,n}$  and  $m_{b,n+1}$  such that  $S_{[c,b]}$  has a fixed point of any size. Hence there exists an  $k$ -desert between these two numbers, where  $k = m_{b,n+1} - M_{b,n} - 1$ .  $\square$

**Theorem 4.3.** *For each  $b \geq 2$  and  $k \in \mathbb{Z}^+$ , there exists an  $k$ -desert base  $b$ .*

*Proof.* Fix  $b \geq 2$  and  $k \in \mathbb{Z}^+$ . Since  $(n - 5/4)(b - 1)^2$  is an increasing linear function of  $n$ , there exists some  $n \geq 2$  such that  $(n - 5/4)(b - 1)^2 \geq k$ . It follows from Lemma 4.2 that there exists an  $k$ -desert base  $b$ .  $\square$

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